Announcements

1) Change zeros on
question 5 a) to ones

Definition: (separated, disconnected)

Two subsets S and T of a metric space X are separated if $\overline{S} \cap T = S \cap \overline{T} = \emptyset$.

A subset $Y \subseteq X$ is disconnected if $\exists\, \underbrace{S, T}_{} \subseteq Y$, S, T separated and nonempty

$S \cup T = Y$ (a separation of Y)

$S \subseteq X$ is connected if S is

not disconnected

Example 1. (separated sets)

$[0,1]=S$ and $[5,10]=T$

are separated since

$S=\overline{S}$, $T=\overline{T}$, and $S\cap T=\emptyset$.

$S=[0,1]$, $T=(1,2)$

S and T are not separated

since $\overline{T}=[1,2]$ and

$S\cap\overline{T}=\{1\}$.

Example 2: ($\mathbb{Q}$) Let's show $\mathbb{Q}$ is disconnected.

Pick $x \in \mathbb{R}$, $x \notin \mathbb{Q}$

$x = \sqrt{2}$

$S = (-\infty, \sqrt{2}) \cap \mathbb{Q}$

$T = (\sqrt{2}, \infty) \cap \mathbb{Q}$

Check that this is a separation!

$\underline{S \cup T = \mathbb{Q}}$ If $y \in \mathbb{Q}$, either

$y > \sqrt{2}$ or $y < \sqrt{2}$.

In the latter, $y \in T$, and

in the former, $y \in S$.

So $S \cup T = \mathbb{Q}$.

$\underline{\overline{S} \cap T = \emptyset}$ $S = (-\infty, \sqrt{2}) \cap \mathbb{Q}$

$\overline{S} = (-\infty, \sqrt{2}]$

since every irrational number

in $(-\infty, \sqrt{2}]$ is a limit point

of S by density of the rationals

If $x > \sqrt{2}$, then

with $\varepsilon = \dfrac{x-\sqrt{2}}{2}$, then

$B(x, \varepsilon) \cap S = \emptyset$, so

$x \notin \overline{S}$.

$\overline{S} = (-\infty, \sqrt{2}]$, then

$$\overline{S} \cap T = (-\infty, \sqrt{2}] \cap ((\sqrt{2}, \infty) \cap \mathbb{Q})$$
$$= \emptyset$$

<u>$\overline{T} \cap S = \emptyset$</u> similar to $\overline{S} \cap T = \emptyset$,

$\overline{T} = [\sqrt{2}, \infty)$

Therefore, $\mathbb{Q}$ is disconnected.

Famous Topological Example

The graph of $f(x) = \sin\left(\frac{1}{x}\right)$, along with the portion of the y-axis $-1 \le y \le 1$, is connected.

Theorem: (connected sets in $\mathbb{R}$)

$S \subseteq \mathbb{R}$ (with the standard metric)

is <u>connected</u> if and only if

whenever $x, y \in S$, $x < y$,

then $\forall$ z with $x < z < y$,

$\boxed{z \in S}$

Proof: $\Rightarrow$) Suppose that S is connected. Let $x, y \in S$, show that $\forall$ $x < z < y$, $z \in S$.

Suppose not Then $\exists$ $x, y \in S$ and a z with $x < z < y$, $z \notin S$.

Then let $T = (-\infty, z) \cap S$ and $V = (z, \infty) \cap S$.

Claim $\{T, V\}$ is a separation for S.

$T \cup V = S$ Just like $\mathbb{Q}$, any $w \in S$ is either bigger than z or less than z

$\overline{T} \cap V = \phi$

$T = (-\infty, z) \cap S$

if $w \in T'$, then $w \in [-\infty, z] = \overline{(-\infty, z)}$

This implies $\overline{T} \subseteq [-\infty, z]$.

Therefore,

$$\overline{T} \cap V \subseteq [-\infty, z] \cap V$$
$$= (-\infty, z] \cap ((z, \infty) \cap S)$$
$$= \phi$$

$\bar{V} \cap T = \emptyset$ Similar,

$\bar{V} \subseteq [z, \infty)$.

Then $\{T, V\}$ is a separation of S. But S is connected, so this is a contradiction.

Therefore, $\forall\, x, y \in S$, if $x < z < y$, $z \in S$.

$\Leftarrow$ Suppose $\forall\, x, y \in S$,

if $x < z < y$, then $z \in S$.

Show: S is connected.

So let T, V be two subsets

of S with $T \cup V = S$.

We'll show: either $\bar{T} \cap V \neq \emptyset$

or $\bar{V} \cap T \neq \emptyset$.

Let $x \in T$, $y \in V$.

If $\forall\, y \in V$ and all $x \in T$

either $x < y$ or $y < x$, then

$\exists\ \alpha \in \mathbb{R}$ such that

$x \leq \alpha \ \forall\ x \in T$

and $y \geq \alpha \ \forall\ y \in V$

(assuming without loss of generality, that $x < y \ \forall\ x \in T,\ y \in V$)

2 cases

$\alpha \notin S$ Choose $x < \alpha$, $x \in T$
$y > \alpha$, $y \in V$.

By our assumption, since
$x < \alpha < y$, we have $\alpha \in S$.
Contradiction, so

$\alpha \in S$ Then $\overline{T} \subseteq (-\infty, \alpha]$.

But if $y \in V$, $y > \alpha$,
then $\forall z$, $\alpha < z < y$,
$z \in S$.

This implies that

$(x, y) \subseteq V \Rightarrow$

$[x, y] \subseteq \overline{V}$.

Since $x \in S$ and $S = T \cup V$

either $x \in T$ or $x \in V$.

If $x \in T$, $x \in T \cap \overline{V}$

If $x \in V$, $x \in V \cap \overline{T}$

Therefore, either $T \cap \overline{V} \neq \emptyset$

or $V \cap \overline{T} \neq \emptyset$.